

On the Uniqueness of the Branching Parameter for a Random Cascade Measure

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An independent random cascade measure μ is specified by a random generator (w_1, \dots, w_c) , $E \sum w_i = 1$ where c is the branching parameter. It is shown under certain restrictions that, if μ has two generators with a.s. positive components, and the ratio $\ln c_1 / \ln c_2$ for their branching parameters is an irrational number, then μ is a Lebesgue measure. In other words, when c is a power of an integer number p and the p is minimal for c , then a cascade measure that has the property of intermittency specifies p uniquely.

KEY WORDS: Random cascades; intermittency; branching parameter.

INTRODUCTION

Intermittency in turbulence is usually expressed in scaling terms of structure functions. For energy $\varepsilon(\Delta_x)$ dissipated in a cell Δ_x , empirical data give

$$\langle \varepsilon(\Delta_x)^q \rangle \sim |\Delta|^{q\tau(q)+d}, \quad |\Delta| \ll 1 \quad (1)$$

where $\langle \cdot \rangle$ denotes spatial averaging, $|\Delta|$ is the linear cell size, and d is the spatial dimension ($d = 1$ in what follows). Intermittency corresponds to scaling exponents $\tau(q)$ of a nonlinear type for $q > 0$. Historically the first interpretation of intermittency is associated with Richardson's idea (see, e.g., ref. 1) as to energy being transmitted from larger to smaller scales in an inertial range (δ, L) : L is the external scale, while $\delta \ll L$ is the Kolmogorov scale at which the dissipation occurs.

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The above idea can be formalized by means of the following recursive procedure which defines an independent random cascade. We denote by $\varepsilon_\alpha^{(n)}$ the energy in a cell $\Delta_\alpha^{(n)}$ of level n . Each cell $\Delta_\alpha^{(n)}$ is divided into c equal subcells of level $(n+1)$; into these the energy $\varepsilon_\alpha^{(n)}$ is transmitted with random coefficients

$$(w_1(\Delta_\alpha^{(n)}), \dots, w_c(\Delta_\alpha^{(n)})) = W(\Delta_\alpha^{(n)}).$$

The vectors $W(\Delta_\alpha^{(n)})$, which are called breakdown coefficients in the physics literature, are statistically independent and identically distributed for different cells $\Delta_\alpha^{(n)}$ of all levels n . Their distribution is specified by the random vector (or *cascade generator*) $W = (w_1, \dots, w_c)$, for which $w_i \geq 0$ and $E \sum w_i = 1$, corresponding to the law of conservation of energy in the average. In what follows we will restrict our consideration to cascade generators for which $P(w_* = 1) < 1$ and $P(w_* > 0) = 1$. Here w_* is the normalized random component of the vector W , namely

$$w_* = \{cw_i \quad \text{with probability } 1/c, \quad i = 1, \dots, c$$

and $Ew_* = 1$.

One can express many cascade properties in terms of w_* . In particular, the following condition: $Ew_* \log_c w_* < 1$ ensures the existence of a non-trivial limit of the measures

$$\varepsilon_n(dx) = \sum_\alpha \varepsilon_\alpha^{(n)} \mathbf{1}_{\Delta_\alpha^{(n)}}(x) dx / |\Delta^{(n)}|$$

as $n \rightarrow \infty$.^(2,3) Following Mandelbrot,⁽⁴⁾ the limiting cascade measure $\varepsilon(dx)$ is considered as a model of dissipated energy field in turbulence.

Under very general conditions the cascade measure $\varepsilon(dx)$ has the intermittency property (1) where \sim denotes logarithmic asymptotics, i.e., $a \sim b$ when $\ln a = \log b(1 + o(1))$ a.s.^(5,6) The scaling exponents in (1) are closely connected with the function

$$\tau^H(q) = q - \log_c Ew_*^q - 1, \quad (2)$$

which represents the heuristic estimate of $\tau(q)$. It is easily found by replacing $\varepsilon(dx)$ with the pre-limit measure $\varepsilon_n(dx)$, $n \gg 1$ and the spatial averaging $\langle \cdot \rangle$ with ensemble averaging, i.e., with the operation of mathematical expectation E . With large $|q|$ these manipulations lead to false estimates of τ .⁽⁵⁾ The true function $\tau(q)$ is identical with (2) in the interval $q \in (q_-, q_+)$, only where $-\infty \leq q_- \leq 0$ and $1 \leq q_+ \leq \infty$. The function τ is linear outside of (q_-, q_+) if $q_- < 0 < 1 < q_+$: $\tau(q) = a_\pm q$. Both lines $a_\pm q$ for finite q_\pm are

tangent to the curve of $\tau^H(q)$, which uniquely specifies the critical points q_- and q_+ as those tangent points closest to 0.

The interpretation of intermittency in terms of cascades uses two assumptions that are not particularly attractive from the standpoint of physics:

- (a) the ratio of adjacent scales $|\Delta_n|/|\Delta_{n+1}|$, i.e., the cascade's branching parameter, is fixed;
- (b) the Kolmogorov dissipation scale is zero: $\delta = 0$.

From (a) it follows that $L/\delta = c^N$, where c and N are integers. Varying L , which is natural for many physical objects, we vary c thereby. For this reason it is desirable to deal with cascades whose statistical properties are independent of the parameter c . This standpoint has proved fruitful for resolving the problem of parametrization of empirical τ -functions. A broad class of functions (2) corresponding infinitely divisible random variables $\log w_*$ was suggested for practical purposes.⁽⁷⁻¹¹⁾ Any τ -function of this type can be produced by a cascade generator of any dimension. Unfortunately, a complete description of τ -functions that would have the above property is unknown.

In some applications there are attempts to introduce a scale densification in the cascade process.^(10, 12) The aim of this modification is twofold: to get rid of the above assumptions (a, b) and to justify a "universal class" of cascades. This idea has unfortunately remained without justification.

An opposite viewpoint on the parameter c for turbulent cascades is contained in ref. 13 where the authors assumed $c = 2$, since the Navier-Stokes equation involved a nonlinearity of the second order. Based on this assumption, the authors derive the statistical conclusion that the coefficients $W(\Delta^{(n)})$ are interdependent for two adjacent levels n and $(n+1)$ in actual turbulence. The conclusion lacks experimental corroboration of the assumed hypothesis $c = 2$. Otherwise it can equally well be regarded as an artefact.

It is our purpose to show that, under conditions that are natural for turbulence, the least integer-valued parameter p in the representation $c = p^n$, $n \geq 1$ is uniquely specified by a cascade measure ($c = p = 2$ in the case of ref. 13). From this it follows that a locally positive cascade measure having the intermittency property and a two generators of significantly different dimensions, i.e., when $\log c_1/\log c_2$ is irrational, does not exist. In other words, the requirement that the cascade measure be independent of the branching parameter is much too fine for the phenomenological model of intermittency. However, if the cascade measure is regarded as the model of a physical object, the above parameter p in $c = p^n$ should have a physical

meaning, hence an algorithm is required to identify it from the cascade measure. Such algorithms are unknown to us.

The present study generalizes the results of my previous work.⁽¹⁴⁾

THE MAIN RESULT

This section consists in the following

Theorem. Suppose a random cascade measure μ on $[0, 1] = I$ is locally positive, i.e., $\mu(\Delta) > 0$ a.s. for any subinterval $\Delta \subset I$, the total mass $M = \mu(I)$ has a second moment, $EM^2 < \infty$, and $q_+ > 2$. If μ has two generators $\xi \in R^{c_1}$ and $\eta \in R^{c_2}$, $0 < c_1 < c_2$, and $\log c_1 / \log c_2$ is irrational, then μ is a Lebesgue measure.

Let us comment on the conditions of this theorem. The main requirements, namely, that μ should be locally positive and $\log c_1 / \log c_2$ should be irrational are essential. For instance, the cascade generator W and the tensor product of its independent copies $W_1 \otimes W_2$ generate the same cascade measure having the branching parameters c and c^2 . A measure of the type $\mu(dx) = \delta(x - \xi) dx$ where ξ is a random uniformly distributed variable on $[0, 1]$ is a cascade measure with a generator of arbitrary dimension: $W = (0, \dots, 0, 1, 0, \dots, 0)$. Here, 1 occupies the i th position with probability $1/c$.

The requirements $EM^2 < \infty$ and $q_+ > 2$ are purely technical in character and are merely needed in the method of proof we employ. Under these conditions $\tau(q) = \tau^H(q)$ for $0 < q < 2$. We remind that the tangent to $\tau^H(q)$ at the point $q_+ < \infty$ passes through $(0, 0)$. To be more specific, the function $-\tau^H(q)$ is convex, so that one should speak of the support line at the point $q_+ < \infty$ rather than of the tangent. Judging by empirical evidence (see, e.g., ref. 1), these requirements do not constitute restrictions on turbulent cascades.

The proof rests on two Lemmas.

Lemma 1. Suppose two vectors $\xi = (\xi_0, \dots, \xi_{c_1-1})$ and $\eta = (\eta_0, \dots, \eta_{c_2-1})$, $1 < c_1 < c_2$ with positive components commute with respect to the tensor product: $\xi \otimes \eta = \eta \otimes \xi$. If $\log c_1 / \log c_2$ is irrational, then both vectors have constant components.

Proof of Lemma 1. We write down the commutation condition for the vectors ξ and η as follows;

$$\eta_{[q]_{c_2}} \xi_{\{q\}_{c_2}} = \xi_{[q]_{c_1}} \eta_{\{q\}_{c_1}}, \quad 0 \leq q < c_1 c_2, \quad (3)$$

where $[q]_n$ and $\{q\}_n$ are the integer part and the remainder resulting from dividing q by n . One has for $q = \alpha < c_1$:

$$\eta_0 \xi_\alpha = \xi_0 \eta_\alpha, \quad 0 \leq \alpha < c_1.$$

Consequently, one can assume $\xi_\alpha = \eta_\alpha$, $\alpha < c_1$ without loss of generality; (3) then becomes

$$\eta_{[q]_{c_2}} \eta_{\{q\}_{c_2}} = \eta_{[q]_{c_1}} \eta_{\{q\}_{c_1}}, \quad 0 \leq q < c_1 c_2. \tag{4}$$

In particular,

$$\eta_{pc_1 + \alpha} = \eta_p \eta_\alpha / \eta_0, \quad 0 \leq pc_1 + \alpha < c_2, \quad 0 \leq \alpha < c_1. \tag{5}$$

Iteration yields

$$\eta_\beta / \eta_0 = \prod_{i=0}^k (\eta_{\alpha_i} / \eta_0), \quad \beta = \alpha_0 + \alpha_1 c_1 + \dots + \alpha_k c_1^k < c_2, \quad 0 \leq \alpha_i < c_1, \tag{6}$$

that is, the vector η can be uniquely reconstructed from the first c_1 coordinates.

Suppose D is the greatest common divisor of c_1, c_2 , and $D := (c_1, c_2) < c_1$. One can then find integer $\alpha_0 < c_1/D := k_1$ and $\beta_0 < c_2/D := k_2$ such that $\alpha_0 c_2 = \beta_0 c_1 + D$.

Denote

$$a_p = \eta_{\beta_0 + p} / \eta_{\alpha_0}, \quad \beta_0 + p < c_2, \\ \vec{\eta}_r = (\eta_{rD}, \eta_{rD+1}, \dots, \eta_{rD+D-1}), \quad 0 \leq r < k_2.$$

Since $\alpha_0 < c_1$, one has $\alpha_0 c_2 + \beta = \beta_0 c_1 + D + \beta < c_1 c_2$ for any $0 \leq \beta < c_2$. Hence, using (3) and the above notation, one has:

$$\eta_\beta = a_{[\beta + D]_{c_1}} \eta_{\{\beta + D\}_{c_1}}, \quad 0 \leq \beta < c_2. \tag{7}$$

From (7) it follows that

$$\vec{\eta}_{pk_1 + i} = \begin{cases} a_p \vec{\eta}_{i+1}, & 0 \leq i < k_1 - 1 \\ a_{p+1} \vec{\eta}_0, & i = k_1 - 1, \end{cases} \tag{8}$$

where $pk_1 + i < k_2$. Put $p = 0$ here. One then arrives at the recurrence relation

$$\vec{\eta}_i = a_0 \vec{\eta}_{i+1}, \quad 0 \leq i < k_1 - 1, \tag{9}$$

whence

$$\vec{\eta}_i = a_0^{-i} \vec{\eta}_0, \quad 0 \leq i < k_1. \quad (10)$$

From (5) one has

$$\vec{\eta}_{pk_1+i} = (\eta_p/\eta_0) \vec{\eta}_i, \quad 0 \leq i < k_1, \quad pk_1+i < k_2. \quad (11)$$

The use of (8), (11), and (9) yields the chain of relations

$$a_p \vec{\eta}_{i+1} \stackrel{(8)}{=} \vec{\eta}_{pk_1+i} \stackrel{(11)}{=} (\eta_p/\eta_0) \vec{\eta}_i \stackrel{(9)}{=} a_0 (\eta_p/\eta_0) \vec{\eta}_{i+1} \quad 0 \leq i < k_1 - 1,$$

where $0 \leq pk_1 + i < k_2$. Put $i = 0$ here. Then

$$a_p/a_0 = \eta_p/\eta_0, \quad 0 \leq pk_1 < k_2. \quad (12)$$

We now make use of (8) with $i = k_1 - 1$:

$$a_{p+1} \vec{\eta}_0 \stackrel{(8)}{=} \vec{\eta}_{pk_1+k_1-1} \stackrel{(11)}{=} (\eta_p/\eta_0) \vec{\eta}_{k_1-1} \stackrel{(10)}{=} (\eta_p/\eta_0) a_0^{-k_1+1} \vec{\eta}_0, \\ 0 \leq pk_1 + k_1 - 1 < k_2,$$

i.e.,

$$a_{p+1}/a_0 = (\eta_p/\eta_0) a_0^{-k_1} \stackrel{(12)}{=} (a_p/a_0) a_0^{-k_1}, \quad 0 \leq pk_1 + k_1 - 1 < k_2,$$

whence

$$a_p/a_0 = a_0^{-pk_1}, \quad 0 \leq p < (k_2+1)/k_1, \quad (13)$$

$$\eta_p/\eta_0 = a_0^{-pk_1}, \quad 0 \leq p < k_2/k_1. \quad (14)$$

From (10), (11), and (14) one has for $r = pk_1 + i < k_2$, $i < k_1$:

$$\vec{\eta}_r \stackrel{(11)}{=} (\eta_p/\eta_0) \vec{\eta}_i \stackrel{(10)}{=} (\eta_p/\eta_0) a_0^{-i} \vec{\eta}_0 \stackrel{(14)}{=} a_0^{-r} \vec{\eta}_0,$$

i.e.,

$$\vec{\eta}_r = a_0^{-r} \vec{\eta}_0, \quad 0 \leq r < k_2. \quad (15)$$

The original relations (4) when expressed in terms of the $\vec{\eta}_i$ have the form

$$\eta_{[r]k_1} \vec{\eta}_{\{r\}k_1} = \eta_{[r]k_2} \vec{\eta}_{\{r\}k_2}, \quad 0 \leq r < c_1 c_2 / D.$$

Hence in virtue of (15) one has

$$\eta_{[r]k_1} a_0^{-\{r\}k_1} = \eta_{[r]k_2} a_0^{-\{r\}k_2}. \tag{16}$$

Let $r = k_2 = pk_1 + i, i < k_1$. Then $p < k_2/k_1$ and

$$\eta_p a_0^{-i} = \eta_1 \tag{17}$$

or

$$a_0^{-pk_1} \stackrel{(14)}{=} \eta_p / \eta_0 \stackrel{(17)}{=} (\eta_1 / \eta_0) a_0^i \stackrel{(16)}{=} a_0^{-k_1+i}.$$

Hence $a_0 = 1$. Otherwise $pk_1 = k_1 - i$ or $k_2 = pk_1 + i = k_1$, which is impossible.

From (16) and $a_0 = 1$ one gets

$$\eta_{[r]k_1} = \eta_{[r]k_2}.$$

One has

$$\eta_p = \eta_0, \quad 0 \leq p < k_2/k_1$$

when $r = pk_1 + i < k_2$ and

$$\eta_p = \eta_1, \quad k_2/k_1 < p < 2k_2/k_1$$

when $k_2 < pk_1 + i < 2k_2$ and so $\eta_p = \eta_0, p < 2k_2/k_1$, since $\eta_1 = \eta_0$.

Proceeding as above, we shall prove in succession that the components of $\vec{\eta}_0$ are constant. By (15) $\vec{\eta}_r = \vec{\eta}_0, 0 \leq r < k_2$, thus we have the desired relation $\eta = (\eta_0, \dots, \eta_0)$.

Consider the case in which c_2 is divisible by c_1 , i.e., $D = c_1$. Put $q = rc_1, r < c_2$ in (4). One gets

$$\eta_{[r]k_2} \eta_{c_1\{r\}k_2} = \eta_r \eta_0 \stackrel{(5)}{=} \eta_{[r]c_1} \eta_{\{r\}c_1}, \tag{18}$$

where $k_2 = c_2/c_1$. However, if c_2 is divisible by c_1 , then by (4),

$$\eta_p = \eta_{pc_1}, \quad 0 \leq p < k_2,$$

i.e., $\eta_{c_1\{r\}k_2} = \eta_{\{r\}k_2}$. Consequently, (18) means that the vectors $\xi = (\eta_0, \eta_1, \dots, \eta_{c_1-1})$ and $\eta = (\eta_0, \dots, \eta_{k_2-1})$ commute with respect to the tensor product. The maximum dimension of the new vectors ξ and η has decreased from c_2 to $\max(c_1, k_2)$. The problem has reduced to the case already considered. The process of reducing the dimension of ξ, η is finite, terminating when the dimensions $c_1^{(k)}$ and $c_2^{(k)}$ are not mutually divisible at

the stage k . Otherwise, as is easily seen, $c_1 = p^{k_1}$ and $c_2 = p^{k_2}$, where p, k_1, k_2 are integers. That case is ruled out, because $\ln c_1 / \ln c_2$ is irrational. When $c_1^{(k)}$ and $c_2^{(k)}$ are not mutually divisible, induction on k applied to (15) will demonstrate that η has constant components. Lemma 1 is proven. ■

Lemma 1 yields an immediate corollary which will be stated here as Lemma 2.

Lemma 2. Let a random cascade measure μ on $I = [0, 1]$ has two random generators $\xi = (\xi_0, \dots, \xi_{c_1-1})$ and $\eta = (\eta_0, \dots, \eta_{c_2-1})$, $0 < c_1 < c_2$, and $\ln c_1 / \ln c_2$ be irrational. If the measure μ is locally positive a.s. and $E[\mu(I)]^\rho < \infty$ for some $\rho > 1$, then $E\xi_\alpha^\rho = E\xi_0^\rho$, $0 \leq \alpha < c_1$ and $E\eta_\beta^\rho = E\eta_0^\rho$, $0 \leq \beta < c_2$. Also,

$$\left[\frac{E\xi_0^\rho}{(E\xi_0)^\rho} \right]^{\frac{1}{\ln c_1}} = \left[\frac{E\eta_0^\rho}{(E\eta_0)^\rho} \right]^{\frac{1}{\ln c_2}}, \tag{19}$$

provided $0 < \rho < q_+$.

Proof of Lemma 2. It follows from the definition of the random cascade measure $\mu^{(4)}$ that it satisfies the following stochastic equation:

$$\mu \stackrel{(d)}{=} \sum z_i \mu_i \circ T_i^{-1} \tag{20}$$

where $z = (z_0, \dots, z_{c_1-1})$ is the generator of μ , the μ_i are independent copies of μ that are statistically independent of z as well, and $T_i x = (i+x)/c$ is a linear mapping of the interval I into $I_i = [i/c, (i+1)/c]$. Note that $\mu \circ T_i^{-1}(I_j) = 0$ for $i \neq j$. The equality $\mu_1 \stackrel{(d)}{=} \mu_2$ for the two measures means that the distributions of $\mu_i(f)$, $i = 1, 2$ are equal for any smooth finite functions $f: R^1 \rightarrow R^1$.

Suppose ξ and η are the generators of μ . Use (20) with weights $z = \xi$, and then use the same representation for each measure μ_i with weights $\eta^{(i)}$. The weights $\eta^{(i)}$ are independent copies of η which are independent of ξ . The result is

$$\mu \stackrel{(d)}{=} \sum_{0 \leq \alpha < c_1} \xi_\alpha \sum_{0 \leq \beta < c_2} \eta_\beta^{(\alpha)} \mu_{\alpha, \beta} \circ T_{\alpha, \beta}^{-1}, \tag{21}$$

where $\mu_{\alpha, \beta}$ are independent copies of μ that are also independent of $\xi, \eta^{(\alpha)}$, $\alpha < c_1$, while $T_{\alpha, \beta}$ is a linear mapping of I into the interval $\delta_{\alpha, \beta} = [c_2\alpha + \beta, c_2\alpha + \beta + 1]/(c_1c_2)$, $0 \leq \alpha < c_1$, $0 \leq \beta < c_2$. Interchanging ξ and η in (21), one gets a representation of μ that involves the random quantities

$\{\tilde{\eta}_{\beta'}\}$ and $\{\tilde{\xi}_{\alpha'}^{(\beta')}\}$ and the intervals $\delta_{\beta', \alpha'} = [c_1\beta' + \alpha', c_1\beta' + \alpha' + 1]/(c_1c_2)$. Obviously, one has $\delta_{\alpha, \beta} = \delta_{\beta', \alpha'}$ when $c_2\alpha + \beta = c_1\beta' + \alpha'$. Consequently,

$$\begin{aligned} E[\mu(\delta_{\alpha, \beta})]^\rho &= E[\xi_\alpha \eta_\beta^{(\alpha)} \mu_{\alpha, \beta}(\delta_{\alpha, \beta})]^\rho = E\xi_\alpha^\rho E\eta_\beta^\rho m_\rho \\ &= E[\eta_{\beta'} \xi_{\alpha'}^{(\beta')} \mu_{\alpha', \beta'}(\delta_{\beta', \alpha'})]^\rho = E\xi_{\alpha'}^\rho E\eta_{\beta'}^\rho m_\rho, \end{aligned} \tag{22}$$

where $m_\rho = E\mu^\rho(I)$. According to ref. 15, the requirement $0 < m_\rho < \infty$, $\rho > 1$ is equivalent to $E\xi_\alpha^\rho < \infty$ ($E\eta_\beta^\rho < \infty$) for all components of the generator. It also follows from (22) that the moments are $E\xi_\alpha^\rho > 0$, because $\mu(\Delta) > 0$ a.s. for any subinterval of I .

Relation (22) means that the vectors $\{E\xi_\alpha^\rho, \alpha = 0, \dots, c_1 - 1\}$ and $\{E\eta_\beta^\rho, \beta = 0, \dots, c_2 - 1\}$ are positive and commute with respect to the tensor product. The use of Lemma 1 therefore yields the right-hand side of Lemma 2: $E\xi_\alpha^\rho = E\xi_0^\rho$, $0 \leq \alpha < c_1$, $E\eta_\beta^\rho = E\eta_0^\rho$, $0 \leq \beta < c_2$. If $0 < \rho < q_+$, one can equate the $\tau^H(\rho)$ for the generators ξ and η . From (2) one has

$$\log_{c_1} E\xi_*^\rho = \log_{c_2} E\eta_*^\rho. \tag{23}$$

But $E\xi_*^\rho = E \sum_\alpha \xi_\alpha^\rho c_1^{\rho-1}$ with $E\xi_\alpha^\rho = E\xi_0^\rho$ and $E\xi_\alpha = E\xi_0 = c^{-1}$. For this reason one has $E\xi_*^\rho = E\xi_0^\rho / (E\xi_0)^\rho$. Similarly, $E\eta_*^\rho = E\eta_0^\rho / (E\eta_0)^\rho$. Substitution of these relations in (23) yields (19). ■

The proof of the theorem uses another obvious number-theoretic fact which will be treated as a separate statement.

Statement 3. Suppose that the integer numbers n_1 and n_2 are not mutually divisible, and that $T\alpha = \{n_2\alpha\}_{n_1}$ where $\{k\}_n$ is the remainder left after dividing k by n . One can then find $0 < \alpha < n_1$ and $k(\alpha) > 0$ such that $T^{k(\alpha)}\alpha = \alpha$.

Proof of the Theorem. We are going to make use of two representations of μ in the form (21). The one is based on the independent generators $\xi, \eta^{(\alpha)}, \alpha = 0, \dots, c_1 - 1$ and the other on η and $\xi^{(\beta)}, \beta = 0, \dots, c_2 - 1$. Consider the values of μ on elements of the partitioning \mathcal{F} of $[0, 1]$ into c_1c_2 equal parts. One then gets equality of distribution for two families of

$$\{\xi_\alpha \eta_\beta^{(\alpha)} M_{\alpha\beta}\} \stackrel{(d)}{=} \{\eta_{\beta'} \xi_{\alpha'}^{(\beta')} M_{\alpha'\beta'}\},$$

where $0 \leq \alpha, \alpha' < c_1, 0 \leq \beta, \beta' < c_2$, and $M_{\alpha\beta}$ are independent copies of $\mu(I)$ under the following correspondence between the subscripts:

$$q(\alpha, \beta) := c_2\alpha + \beta = c_1\beta' + \alpha'.$$

Here, q , $0 \leq q < c_1 c_2$ is the natural numbering of elements of the partitioning \mathcal{F} on $[0, 1]$. As follows from Lemma 2, the moments

$$E[\zeta_\alpha \eta_\beta^{(\alpha)} M_{\alpha\beta}]^\rho = E \zeta_0^\rho E \eta_0^\rho m_\rho, \quad \rho = 1, 2,$$

are independent of α, β . We shall make use of that circumstance for calculating the moments $E\mu(\delta_q) \mu(\delta_{q+1})$ where δ_q is an element of \mathcal{F} with index q . The twofold representation (21) for μ yields a set of equations. We want to write it down in compact form by first denoting

$$\begin{aligned} a_\alpha &= E \zeta_{\alpha-1} \zeta_\alpha / E \zeta_0^2, & b_\beta &= E \eta_{\beta-1} \eta_\beta / (E \eta_0)^2 \\ \vec{a} &= (a_1, \dots, a_{c_1-1}), & \vec{b} &= (b_1, \dots, b_{c_2-1}) \\ V_a &= E \zeta_0^2 / (E \zeta_0)^2, & V_b &= E \eta_0^2 / (E \eta_0)^2. \end{aligned}$$

Now note that

$$E \eta_{\beta_1}^{(\alpha_1)} \eta_{\beta_2}^{(\alpha_2)} = \begin{cases} (E \eta_0)^2, & \alpha_1 \neq \alpha_2 \\ E \eta_{\beta_1} \eta_{\beta_2}, & \alpha_1 = \alpha_2. \end{cases}$$

Similar equalities also hold for $\zeta_\alpha^{(\beta)}$. Consequently, the equations derived by calculating the moments $E\mu(\delta_q) \mu(\delta_{q+1})$ in two different ways have the form $X = Y$ where

$$\begin{aligned} X &= \{V_b \vec{a}; b_1, V_b \vec{a}; b_2, V_b \vec{a}; \dots; b_{c_2-1}, V_b \vec{a}\}, \\ Y &= \{V_a \vec{b}; a_1, V_a \vec{b}; a_2, V_a \vec{b}; \dots; a_{c_1-1}, V_a \vec{b}\}. \end{aligned}$$

It is our aim to show that the relation $X = Y$ yields $V_a = E \zeta_0^2 / (E \zeta_0)^2 = 1$, i.e., the variance of ζ_0 , hence that of ζ_α , $0 \leq \alpha < c_1$, equals zero. Consequently, the generator ζ has identical nonrandom components, so that μ is a Lebesgue measure.

From the fact that the first $c_1 - 1$ coordinates of X and Y are equal one has

$$V_b a_i = V_a b_i, \quad 1 \leq i < c_1.$$

For this reason the coordinates of the vectors X and Y in the notation $V_a b_i = \tilde{b}_i$ have the form

$$X_r = \begin{cases} \tilde{b}_{\{r\}_{c_1}}, & \{r\}_{c_1} \neq 0 & \text{(a)} \\ V_a^{-1} \tilde{b}_{[r]_{c_1}}, & \{r\}_{c_1} = 0 & \text{(b)} \end{cases} \quad (24)$$

$$Y_r = \begin{cases} \tilde{b}_{\{r\}_{c_2}}, & \{r\}_{c_2} \neq 0 & \text{(a)} \\ V_b^{-1} \tilde{b}_{[r]_{c_2}}, & \{r\}_{c_2} = 0 & \text{(b)}. \end{cases} \quad (25)$$

Putting $r = c_1 k$, one derives

$$b_k = \begin{cases} Vb_{\{c_1 k\}_{c_2}}, & \{c_1 k\}_{c_2} \neq 0 \\ Ub_{c_1 k/c_2}, & \{c_1 k\}_{c_2} = 0 \end{cases} \quad (26)$$

where $V = V_a$, $U = V_a/V_b$. Relation (26) also holds for the b_k and \tilde{b}_k .

Let $c_2 = c_1^{r_1} c'_2$ where $r_1 \geq 0$ is the maximum multiplicity of c_1 in c_2 .

Consider the case $(c'_2, c_1) < \min(c'_2, c_1)$.

Here, $c'_2 > 1$, because $\ln c_2 / \ln c_1$ is irrational. Put $k = c_1^{r_1} p$, $p < c'_2$ in (26). Then

$$b_{c_1^{r_1} p} = Vb_{c_1^{r_1} \{c_1 p\}_{c'_2}}, \quad 1 \leq p < c'_2. \quad (27)$$

According to Statement 3 for the operation $Tp = \{c_1 p\}_{c'_2}$, one finds $1 \leq p_0 < c'_2$ and $k_0 = k(p_0) > 1$ such that $T^{k_0} p_0 = p_0$. Iteration of (27) then yields

$$b_{c_1^{r_1} p_0} = V^{k_0} b_{c_1^{r_1} p_0}.$$

One has $b_\alpha > 0$. Hence $V = 1$, which is the desired result.

One is now entitled to assume that $c_2 = c_1^{r_1} c'_2$ and $r_1 > 0$. In that case the equations $X = Y$ are equivalent to

$$b_p = \begin{cases} b_{\{p\}_{c_1}}, & \text{(a)} \\ V^{-1}b_{p/c_1}, & \text{(b)} \\ b_{\{p\}_{c_2/c_1}}, & \text{(c)} \\ Ub_{p/(c_2/c_1)}, & \text{(d)} \end{cases} \quad (28)$$

where $1 \leq p < c_2$.

This can be seen as follows. Equation (28a) can be derived by comparing (24a) and (25a); (28b) results from a comparison between (24b) and (25b); and (28d) from that between (24b) and (25b). A few words are required to explain (28c). From (24b) and (25a) one has

$$V^{-1}b_\beta = b_{\{\beta c_1\}_{c_2}} = b_{c_1 \{\beta\}_{(c_2/c_1)}} = V^{-1}b_{\{\beta\}_{(c_2/c_1)}}.$$

The last equality in the above sequence follows from the first.

Consider the case $(c'_2, c_1) = c'_2$.

The right-hand side can not be equal to c_1 , otherwise c'_2 would be divisible by c_1 , and the number r_1 in the representation $c_2 = c_1^{r_1} c'_2$ will not be the maximum multiplicity of c_1 in c_2 .

To sum up, one has $c_2 = c_1^{r_1} c_2'$, $r_1 \geq 1$, and $c_1 = (c_2')^{s_1} c_1'$, $s_1 \geq 1$ where s_1 is the maximum multiplicity of c_2' in c_1 . Note that $c_1' > 1$, otherwise $\ln c_1 / \ln c_2$ would be rational. From (28) one has

$$Ub_\alpha \stackrel{(28d)}{=} b_{\alpha c_1^{r_1-1} c_2'} \stackrel{(28b)}{=} V^{1-r_1} b_{\alpha c_2'} \stackrel{(28a)}{=} V^{1-r_1} b_{\{\alpha c_2'\}_{c_1}}. \tag{29}$$

If $r_1 = 1$, then $c_2' = c_2 / c_1$. Therefore,

$$b_\alpha \stackrel{(28b)}{=} V b_{\alpha c_1} = V b_{\alpha (c_2')^{s_1} c_1'} \stackrel{(28c)}{=} V b_{\alpha c_1'} \stackrel{(28d)}{=} V b_{\{\alpha c_1'\}_{c_2'}}. \tag{30}$$

Let $(c_1', c_2') < \min(c_1', c_2')$. In that case (30) will yield $V = 1$, similarly to the above argument.

It can now be assumed that $r_1 > 1$. Put $\alpha = (c_2')^{s_1} \alpha'$ in (29). One then has from (29):

$$Ub_{(c_2')^{s_1} \alpha'} = V^{1-r_1} b_{(c_2')^{s_1} \{\alpha' c_2'\}_{c_1'}}.$$

When $(c_2', c_1') < \min(c_2', c_1')$, the standard procedure would yield $U = V^{1-r_1}$ or $V_b = V_a^{r_1}$. However, according to Lemma 2,

$$V_a^{\ln c_2 / \ln c_1} = V_b. \tag{31}$$

When $V_a \neq 1$, one has $\ln c_2 / \ln c_1 = r_1$, which is impossible. Therefore, one has $V_a = 1$ for the case $r_1 \geq 1$ as well, as was to be proved.

We have arrived at the situation in which $c_2 = c_1^{r_1} c_2'$, $r \geq 1$, $c_1 = (c_2')^{s_1} c_1'$ and c_1' is divisible by c_2' . We will show that a set of equations can be written down that is similar to (28), but where c_1 and c_2 have been replaced with c_1' and c_2' . Also, the coefficients of U, V which have the form $V_a^{k_1} V_b^{k_2}$ with k_i integer will be replaced with similar coefficients. Consequently, the proof will reduce to the preceding with smaller c_i . The reduction process is finite. It will terminate, when c_1' is no longer divisible by c_2' . Otherwise $\log c_2 / \log c_1$ will be a rational number.

It remains to find the analog of (28). One has

$$Ub_q \stackrel{(28d)}{=} b_{c_2 q / c_1} = b_{c_1^{r_1-1} c_2' q} \stackrel{(28b)}{=} (V^{-1})^{r_1-1} b_{c_2' q}.$$

Hence

$$b_p = \tilde{U} b_{p/c_2'}, \quad p < c_1 c_2', \tag{32}$$

where $\tilde{U} = UV^{r_1-1}$. Further,

$$b_q \stackrel{(28b)}{=} V b_{c_1 q} = V b_{(c_2')^{s_1} c_1' q} \stackrel{(32)}{=} V \tilde{U}^{s_1} b_{c_1' q},$$

which gives the analog of (28b):

$$b_p = V_1^{-1} b_{p/c'_1}, \quad p < c'_1 c'_2, \tag{33}$$

where $V_1 = V \tilde{U}^{s_1}$.

From (32, 33) one derives the analog of (28d) with a new coefficient $U_1 = \tilde{U} V_1$:

$$\tilde{U} b_{p/(c'_2/c'_1)} \stackrel{(32)}{=} b_{c'_1 p} \stackrel{(33)}{=} V_1^{-1} b_p, \quad p < c'_2. \tag{34}$$

We now are going to derive the analogs of (28a) and (28c). From (28a) one has

$$b_{(c'_2)^{s_1} p'} = b_{(c'_2)^{s_1} \{p'\}_{c'_1}}, \quad p' < c'_2 c'_1.$$

The use of (32) on both sides of the above equality will yield

$$b_{p'} = b_{\{p'\}_{c'_1}}, \quad p' < c'_2 c'_1. \tag{35}$$

Similarly, (28c) yields

$$b_p = b_{\{p\}_{c_1^{r_1-1} c_2}}.$$

Substituting $p = c_1^{r_1-1} p' < c_2$ and using (28b), one gets

$$b_{p'} = b_{\{p'\}_{c_2}}, \quad p' < c_1 c'_2.$$

Hence one gets for $p' = c'_1 q, q < c'_2$ with the help of (33):

$$b_q = b_{\{q\}_{(c'_2/c'_1)}}. \tag{36}$$

Relations (33)–(36) are the analog of (28). The proof of the theorem is complete.

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